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Infrared bound for the massless propagator in a Yang–Mills field

R F Streater
Bedford College, University of London, Regent's Park, London NW1 4NS, UK

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Abstract. We consider the covariant Laplacian in $\mathbb{R}^n$ with Dirichlet boundary conditions on the boundary of a 'regular' region Λ, in an arbitrary Yang–Mills field A of class C^1. We prove that its Green function $G_\Lambda(A)$ obeys

$$\|G_\Lambda(A)\| \leq \|G_\Lambda(0)\| \qquad \text{for all } A.$$

The proof is based on a comparison theorem with finite-difference operators, and a result for gauge fields on a lattice.

1. Introduction and definitions

In an earlier paper (Streater 1980) we obtained a lower bound for the finite-difference Laplacian with Dirichlet boundary conditions, in terms of a purely geometrical size, $d(\Lambda)$ of the region Λ, $\Lambda \subseteq \mathbb{R}^2$. Weinberger's inequality (Weinberger 1956, see also Wasow and Forsythe 1960), which relates the eigenvalues of the Laplacian in a region Λ to those of the finite-difference Laplacian in a bigger grid $\Lambda^* \supseteq \Lambda$, was then used to obtain a lower bound for the Laplacian itself. This gives a slight improvement on Hayman's result (Hayman 1978). In this paper we show that the same bound holds, too, for the covariant Laplacian coupled to an arbitrary gauge field A belonging to any unitary representation of any Lie group $\mathcal{G}$ (the gauge group). The main tool is a comparison theorem with the analogous finite-difference operator, similar to Weinberger's inequality. The method gives similar estimates in higher dimensions.

We now briefly describe the Yang–Mills theory in Euclidean space and its version on a lattice. We are given a Lie group $\mathcal{G}$ and to each pair of points x, y in $\mathbb{R}^n$ and (rectifiable) path l from x to y, is associated a group element $g(x, y; l)$. This is postulated to obey

(i) $\qquad g(x, x; 0) = 1_{\mathcal{G}}$,

(ii) $\qquad g(x, y; l) = g^{-1}(y, x; -l)$,

(iii) $\qquad g(x, y; l)g(y, z; l') = g(x, z; l \cup l')$,

where $1_{\mathcal{G}}$ is the identity in $\mathcal{G}$ and $l \cup l'$ is the obvious continuous path from x to z. Smoothness conditions will be imposed on g as needed. We postulate that $\mathcal{G}$ is continuously represented by unitary operators in a Hilbert space L. The vectors in L describe the internal degrees of freedom of a multiplet of particles, whose complete description is by a vector in $K = L \otimes L^2(\mathbb{R}^n)$. The *connection* between the wavefunctions at different points along the path l is defined to be

$$V(x, y; l) = V(g(x, y; l)). \tag{1}$$

To each $x \in \mathbb{R}^n$, and unit vector $\hat{n} \in \mathbb{R}^n$, the map

$$\lambda \mapsto g(x, x + \lambda\hat{n}; l), \qquad \lambda \in \mathbb{R}, \tag{2}$$

(where l is the straight line from x to $x + \lambda\hat{n}$) defines a curve in G. If g is a c^1-function of y, then this curve (which passes through $1_{\mathscr{G}}$ when $\lambda = 0$) has a unique tangent vector $a_{\hat{n}}(x)$ at $\lambda = 0$, $a_{\hat{n}}(x)$ lying in the Lie algebra $\mathrm{d}\mathscr{G}$ and being a continuous function of x. The *gauge field* at x, $A_{\hat{n}}(x)$, in the direction $\hat{n}$ is the representation of $a_{\hat{n}}(x)$ coming from V; thus

$$\mathrm{i}A_{\hat{n}}(x) = \mathrm{d}V(a_{\hat{n}}(x)) = \lim_{\lambda\to 0} \frac{V(x, x+\lambda\hat{n}; l) - 1}{\lambda}.$$

Thus for each direction $\hat{n}$, $A_{\hat{n}}(x)$ is a self-adjoint operator on L representing the Lie algebra $\mathrm{d}\mathscr{G}$. It defines a self-adjoint operator on $K = L \otimes L^2(\mathbb{R}^n)$, where the x in $A_{\hat{n}}(x)$ is 'multiplication by x' on $L^2(\mathbb{R}^n)$. The covariant derivative of a vector field $\psi \in K$ is

$$\begin{aligned}\nabla_j\psi &= \lim_{\lambda\to 0} \left(V(x, x+\lambda\hat{j}; l)\psi(x+\lambda\hat{j}) - \psi(x)\right)\lambda^{-1} \\ &= \left(\mathrm{i}A_j(x) + \frac{\partial}{\partial x_j}\right)\psi,\end{aligned} \tag{3}$$

where $\hat{j}$ is taken along the j-direction. The covariant Laplacian is defined to be

$$-\Delta_\Lambda(A) = \sum_{j=1}^{n} \nabla_j^*\nabla_j = \sum_{j=1}^{n} \left(A_j + \mathrm{i}\frac{\partial}{\partial x_j}\right)^2, \qquad \text{(Dirichlet)}, \tag{4}$$

acting on the multiplet states in $K(\Lambda) \equiv L \otimes L^2(\Lambda)$. The covariant massless propagator, for which we seek an estimate, is

$$G_\Lambda(A) = (-\Delta_\Lambda(A))^{-1}. \tag{5}$$

The lattice version of a gauge theory replaces $\mathbb{R}^n$ by $\mathbb{Z}^n$ and restricts the possible paths to the union of bonds i.e. links between the lattice sites. A wavefunction of the multiplet is taken to belong to $l^2(\mathbb{Z}^n, L)$. The gauge field is introduced by defining, for each bond in $\mathbb{Z}^n$ (i.e. each ordered pair $(x, y) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with x, y nearest neighbours), a group element $g(x, y) \in \mathscr{G}$. From this we can construct the group element connecting x_1 to x_m by the chain $(x_1, \ldots, x_m) = l$ to be

$$g(x_1, x_m; l) = g(x_1, x_2)g(x_2, x_3) \ldots g(x_{m-1}, x_m).$$

This definition ensures that g obeys the discrete analogue of (iii). In what follows, only the concept of $g(x, y)$ for nearest neighbours will be needed. The discrete analogue of the covariant derivative (3) is

$$(\delta_j\psi)(x) = h^{-1}(V(x, x+h\hat{j})\psi(x+h\hat{j}) - \psi(x)). \tag{6}$$

Here $V(x, y) = V(g(x, y))$ is the given representation on L, and $\hat{j} \in \mathbb{Z}^n$ is one of $(1, 0, \ldots, 0), (0, 1, \ldots, 0), (0, \ldots, 0, 1)$ where h is the mesh size. The analogue of the covariant Laplacian is

$$-D(g)=\sum_{j=1}^{n}\delta_j^*\delta_j. \tag{7}$$

In considering Dirichlet boundary conditions for $D(g)$ in a region Λ we must note that neither δ_j nor δ_j^* leaves $l^2(\Lambda; L)$ invariant. As in Streater (1979) we introduce the projection E_Λ from $l^2(\mathbb{Z}^n, L)$ to $l^2(\Lambda, L)$ and define the Dirichlet difference Laplacian to be

$$D_\Lambda(g)=E_\Lambda D(g)E_\Lambda. \tag{8}$$

In the next section we show that the lowest point in the spectrum of $-\Delta_\Lambda(A)$ is greater than the lowest point in the spectrum of $-D_{\Lambda^*}(g)$ for a certain field g, where $\Lambda^*\subseteq\mathbb{Z}^n$ is slightly larger than $\Lambda\subseteq\mathbb{R}^n$. The method is adapted from Weinberger (1956, see also Wasow and Forsythe 1960).

2. The comparison theorem

Consider the operator δ_j (equation (6)) on a mesh of size h, embedded in $\mathbb{R}^n$ on which the smooth gauge field $A_j(x)$ is defined. We would like to write (6) as the integral of its differential from x to $x+j$ along the bond. To this end define, for each $\psi\in l^2(\mathbb{Z}^n, L)$, the one-parameter family of vectors

$$\Phi(\gamma)=V(x, x+\gamma\hat{j})\psi(x+\gamma\hat{j}), \qquad 0\leq\gamma\leq h.$$

Then

$$\begin{aligned}\Phi(\gamma+\gamma')&=V(x, x+\gamma\hat{j}+\gamma'\hat{j})\psi(x+\gamma\hat{j}+\gamma'\hat{j})\\&=V(x, x+\gamma\hat{j})V(x+\gamma\hat{j}, x+\gamma\hat{j}+\gamma'\hat{j})\psi(x+\gamma\hat{j}+\gamma'\hat{j}).\end{aligned}$$

Thus

$$\begin{aligned}&\gamma'^{-1}(\Phi(\gamma+\gamma')-\Phi(\gamma))\\&\quad=V(x, x+\gamma\hat{j})\frac{V(x+\gamma\hat{j}, x+\gamma\hat{j}+\gamma'\hat{j})-1}{\gamma'}\psi(x+\gamma\hat{j}+\gamma'\hat{j})\\&\qquad+V(x, x+\gamma\hat{j})(\psi(x+\gamma\hat{j}+\gamma'\hat{j})-\psi(x+\gamma\hat{j}))\gamma'^{-1}.\end{aligned}$$

Take the limit $\gamma'\to0$ giving

$$\frac{\partial\Phi(\gamma)}{\partial\gamma}=V(x, x+\gamma\hat{j})\left(\mathrm{i}A_j(x+\gamma\hat{j})+\frac{\partial}{\partial x_j}\right)\psi(x+\gamma\hat{j}).$$

We conclude that

$$\begin{aligned}h\delta_j\psi&=\int_0^h V(x, x+\gamma\hat{j})\left(\mathrm{i}A_j(x+\gamma\hat{j})+\frac{\partial}{\partial x_j}\right)\mathrm{d}\gamma\psi(x+\gamma\hat{j})\\&=\int_0^h V(x, x+\gamma\hat{j})\nabla_j\psi(x+\gamma\hat{j})\,\mathrm{d}\gamma.\end{aligned}$$

Lemma

$$\|\delta_j\psi\|^2\leq h^{-1}\int_0^h \mathrm{d}\gamma\|\nabla_j\psi(x+\gamma\hat{j})\|^2.$$

Proof

$$h\delta_j\psi = V(x, x+h\hat{j})\psi(x+h\hat{j}) - \psi(x)$$
$$= \int_0^h \mathrm{d}\gamma V(x, x+\gamma\hat{j})\nabla_j\psi(x+\gamma\hat{j}).$$

Let $\phi^k(\gamma)$ be the kth component of $V(x, x+\gamma\hat{j})\nabla_j\psi(x+\gamma\hat{j})$, $1 \leq k \leq \dim L$. Then

$$h^2\|\delta_j\psi\|^2 = \sum_k \left|\int_0^h \mathrm{d}\gamma\phi^k(\gamma)\right|^2 = \sum_k |\langle 1, \phi^k(\gamma)\rangle_{L^2(0,h)}|^2$$
$$\leq \sum_k \left[\left(\int_0^h 1^2\,\mathrm{d}\gamma\right)\int \mathrm{d}\gamma|\phi^k(\gamma)|^2\right] = h\int \mathrm{d}\gamma \sum_k |\phi^k(\gamma)|^2$$
$$= h\int_0^h \mathrm{d}\gamma\|V(x, x+\gamma\hat{j})\nabla_j\psi(x+\gamma\hat{j})\|^2$$
$$= h\int_0^h \|\nabla_j\psi(x+\gamma\hat{j})\|^2\,\mathrm{d}\gamma. \qquad \square$$

Now let $\Lambda \subseteq \mathbb{R}^n$ be any open set and let $\Lambda^\times \subseteq \mathbb{Z}^n$ be the lattice of mesh h such that $x \in \Lambda^\times$ if and only if $x+\alpha \in \Lambda$ for some α with $|\alpha_j| \leq h$, $j = 1, 2, \ldots, n$. Let Λ^* denote the lattice $\Lambda^\times$ together with its nearest neighbours. Clearly $\Lambda^* \supseteq \Lambda \cap \mathbb{Z}^n$ and Λ^* contains an extra row on the 'left-hand side' of Λ in each coordinate. If X is an operator, we denote its spectrum by $\sigma(X)$. Let $V(x, y)$ be a gauge potential of class $C^1(\Lambda)$, and for each $\alpha \in \mathbb{R}^n$ with $0 \leq \alpha_j \leq h$ let $V_\alpha(x, y) = V(x+\alpha, y+\alpha)$. Let

$$\lambda = \inf \sigma(-\Delta_\Lambda(V))$$
$$\lambda^*_\alpha = \inf \sigma(-D_{\Lambda^*}(V_\alpha))$$
$$\lambda^* = \inf_\alpha \lambda^*_\alpha.$$

Theorem. $\lambda \geq \lambda^*$ (the comparison theorem).

Proof. Let $\psi \in C_0^\infty(\Lambda, L)$, i.e. each component of ψ vanishes in the neighbourhood of the boundary of Λ and supp $\psi \subset \Lambda$. Because of the choice of Λ^*, $\phi(x; \alpha) \equiv \psi(x+\alpha)$, defined on the lattice Λ^*, obeys the boundary conditions for the Dirichlet difference Laplacian in Λ^*. Hence it is a trial function for the Rayleigh–Ritz inequality, with lattice gauge field V_α:

$$\lambda^*_\alpha\|\phi\|^2 \leq \sum_{j=1}^n \|\delta_j^\alpha\phi\|^2$$

giving

$$h^2\lambda^*_\alpha \sum_x \|\psi(x+\alpha)\|_L^2 \leq \sum_x \sum_{j=1}^n \|V(x+\alpha, x+\alpha+\hat{j})\psi(x+\alpha+\hat{j}) - \psi(x+\alpha)\|_L^2.$$

This is true for all α, $0 \leq \alpha_j \leq h$. Replace λ^*_α on the left by λ^* (the inequality clearly remains true) and integrate over $0 \leqslant \alpha_j \leqslant h$, $j = 1, \ldots, n$. This gives

$$h^2\lambda^* \sum_x \int_0^h d\alpha_1 \ldots \int_0^h d\alpha_n \|\psi(x+\alpha)\|_L^2$$
$$\leqslant \int d^n\alpha \sum_x \sum_{j=1}^n \|V(x+\alpha, x+\alpha+\hat{j})\psi(x+\alpha+\hat{j}) - \psi(x+\alpha)\|_L^2.$$

The left-hand side is $\lambda^* h^2 \int_\Lambda \|\psi\|_L^2 \, d^n x$. The typical term on the right-hand side is

$$\int d^n\alpha \sum_x \|V(x+\alpha, x+\alpha+\hat{j})\psi(x+\alpha+\hat{j}) - \psi(x+\alpha)\|_L^2$$
$$\leqslant \int_0^h d^n\alpha \sum_x h \int_0^h d\gamma \|\nabla_j\psi(x+\alpha+\gamma\hat{j})\|_L^2$$

by the lemma; this is equal to

$$h \int d^n x \int_0^h \|\nabla_j\psi(x+\gamma\hat{j})\|_L^2 \, d\gamma.$$

But dx is translation invariant, so this is equal to

$$h\int_0^h d\gamma \int d^n x \|\nabla_j(x+\gamma\hat{j})\|_L^2 = h\int_0^h d\gamma \int d^n x \|\nabla_j\psi(x)\|_L^2 = h^2 \int_\Lambda d^n x \|\nabla_j\psi\|_L^2.$$

Summing over j we get

$$\lambda^* \int_\Lambda \|\psi\|^2 \, d^n x \leqslant \sum_j \int_\Lambda d^n x \|\nabla_j \psi\|_L^2$$

so

$$\lambda^*(A) \leqslant \frac{\langle\psi, -\Delta_\Lambda(A)\psi\rangle}{\langle\psi, \psi\rangle} \qquad \text{for all } \psi \in C_0^\infty(\Lambda, L).$$

Take the supremum over the right-hand side. This gives $\lambda^*(A) \leqslant \lambda(A)$.

3. The infrared bound

Brydges *et al* (1980) have proved that the Green function for a lattice gauge theory is pointwise dominated by the propagator with $V = 1$ for the same region. This implies that $\lambda^*(V) \geqslant \lambda^*(1)$ for any gauge field. Combining with our result in § 2, we see that $\lambda_\Lambda(V) \geqslant \lambda^*(1)$ for every lattice $\Lambda^* \supseteq \Lambda$. For regular regions (without too many spikes) we have (Weinberger 1956, see also Wasow and Forsythe 1960)

$$\lambda_\Lambda = \sup_{\Lambda^* \supseteq \Lambda} \lambda_\Lambda^*(1)$$

so we obtain

$$\lambda_\Lambda(V) \geqslant \lambda_\Lambda(1).$$

That is, the lowest eigenvalue (or lowest point on the spectrum if Λ is not compact) of the covariant Laplacian is bounded below by that of the ordinary Dirichlet Laplacian for the same region. I have obtained a geometrical bound for $\lambda^*(1)$ if $n=2$ or if Λ is convex (Streater 1980). If $n=2$ we got

$$\lambda^* \geqslant \frac{2}{9h^2}\left[1-\cos\left(\frac{\pi}{8d/h+8}\right)\right],$$

where d is the size of the largest square that can be drawn inside Λ^*. By varying Λ^* and reducing h we can optimise this estimate. Combining with the main result of § 2, we obtain

$$\lambda(V) \geqslant \sup_{\Lambda^* \supset \Lambda} \frac{2}{9h^2}\left[1-\cos\left(\frac{\pi}{8d/h+8}\right)\right].$$

Actually, this result can be obtained directly by the methods of Streater (1980) without appealing to Brydges *et al* (1980). Taking the limit $h \to 0$ gives

$$\lambda(V) \geqslant \frac{\pi^2}{576d^2}.$$

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